A New Entropy Probability Model of Trip Distribution

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(Received 13 January 2019; Revised 27 June 2019; Accepted 17 July 2019)

Abstract: In the traditional entropy maximising derivation of the gravity model, any micro-distribution of the individual flows within a linkage is seen as equally likely a priori, and the aim is to find the macro-distribution flows (or linkage trips) which is most probable, subject to relevant constraints. The result of an equal likelihood assumption is that the probability of trips from zone i to j is always the number of trips from zone i to j divided by the total number of trips in the study area. With unequal or unknown likelihood of trip-making within the various origins-destination linkages, the probabilities will lead to different quantities. A method has been proposed, using the principle of maximum entropy, to derive the trip probability distribution catering for the biases within the linkages. The result is a specified model with two deterrent factors, namely, a trip frequency function and a trip cost function. Calibration is performed using two criteria. The cumbersome balancing mechanism of the traditional spatial interaction model is no longer present. These two criteria are obtained directly from the principle of maximum entropy. The first criterion is the natural constraint equation. The other criterion is the selection of the probability distribution with the highest entropy value, which represents the most unbiased, most random and most objective distribution consistent with the information contained in the form of constraints. The principle of maximum entropy thus provides its own calibration criteria, and avoids the use of any inherently biased statistic. Since two calibration criteria are used, the possibility of many combinations of multi-parameters giving optimum fit, as with a single calibration statistic, is significantly reduced, if not altogether eliminated. The strength of this model is its forecasting ability. This paper describes a forecasting approach with examples, and provides a comparison with the highly accepted gravity model.

Keywords: Origin-Destination Matrix Estimation; Maximum Entropy Solution; Goodness-of-Fit; Trip Residual Patterns

1. Introduction

Spatial interaction models as used in travel demand analysis are more commonly called trip distribution models, since they are used for the analysis of trip interchange in a four-stage urban transportation modelling process. This modelling process forms a major part of the urban transportation planning process. Of the four (4) sub-model stages, the trip distribution sub-model has received the most attention by researchers, as this attempts to capture the travel pattern and thus simulate the spatial structure of the study area. Trip distribution is an invaluable portion of the transportation planning process since it leads to the creation of origin-destination (OD) matrices (Jin et al., 2014). The adequacies of the other stages hinge upon proper specifying of the spatial structure. The technique most widely used for this stage is the gravity model.

The trip distribution computation is more popularly called the OD matrix estimation, and is still considered one of the key components in traffic planning and traffic control (He and Dai, 2011). Researchers are interested in developing improved OD matrix estimation results. For example in recent times, Zhao et al. (2018) recommended a multiclass probit-based OD model using multiple data types; Liu et al (2018) considered OD demand forecasting for a large-scale bike sharing system; Kattan and Abdulhai (2011) evaluated the performance of comparative analysis of evolutionary, local search, and hybrid approaches to OD traffic estimation; Goel et al. (2006) developed a Bayesian methodology for estimating state-wide annually averaged OD flows using link-level annual average daily traffic estimates obtained from remote sensed imagery as well as from traditional ground counts; Wang et al. (2010) provided analytical approach to describe distribution of traffic entering intersections on a grid road network without the need for using a detailed OD matrix; and Guler (2014) developed a travel demand modelling framework to calculate the transportation demand of the Marmaray corridor, as well as a model to estimate the OD matrices. Moreover, Xian et al. (2012) discussed an improved distribution model in strategic transport model of Beijing, and Li and Chen (2011) presented a transit OD matrix estimation method based on population and land use.

Transportation planning and traffic micro-simulation software must include a module consisting of OD matrix estimation, or a trip distribution model. This model can be simplistic, such as growth factor methods or sophisticated, such as the gravity model with many of its variations.

Trinidad and Tobago (T&T) has had three national transportation studies. These were: Parker and Parsons-Brinckerhoff (1966) National Transportation Plan; Trinidad and Tobago, Government of Trinidad and Tobago; N.D. Lea – Planning Associates Ltd. (1983) National Transportation Policy Project; Government of the Republic of Trinidad and Tobago; and a study
The datasets used in this paper is for the island of Trinidad collected in 1981 and in 2006. The OD person trip movements for an average weekday, were aggregated to a system of 14 zones, known as Regional Corporations, with the zonal person trip ends ranging between 20,000 and 115,000 in 2006, with an average zonal person trip end of 68,000. The shortest path time indices were produced using the network data for both 1981 and 2006. Intrazonal trips were not modelled, and therefore, have been excluded from all analysis.

2. Trip Probability Distribution Derived

The Principle of Maximum Entropy states that “...when we have only partial information about the probabilities, we should choose the probabilities, so as to maximise the uncertainty about the missing information. According to this principle, we should use all information we have and scrupulously avoid using any information not given to us. We should be as uncommitted as possible about the missing information. Since entropy is also a measure of randomness, according to this principle, we choose the most random distribution subject to the satisfaction of the given constraints.” (Kapur and Kesavan, 1987, p.25).

The maximum entropy solution is not a unique one (Jaynes, 1979), but is subject to one's choice of constraint, which reflects the actual knowledge one has about the system. On the contrary, it is deceptively easy to use the principle in the sense that any redundant constraint is eliminated in the maximisation process. Thus, in order to get an accurate description, we need to recognise all the uncontrollable system influences through inspection of the data structure, without assuming any information not given, while ensuring that all the mathematical rules of probability theory are strictly obeyed.

While some may argue that the cost constraint is a convenience and not an actual constraint, it should be stated that the format of the constraint equation under information theoretic entropy maximisation is such that (a) only available information is utilised; (b) the constraint equation is linear; and (c) the simplest form is used based on trial and error and/or experience. The cost constraint has been chosen in accordance with these fundamental rules. So, in effect, it may be a convenience, but is fully within the rules. It is possible that other forms of the constraint equation may be found suitable.

For the base year we are usually supplied with the trip matrix \( \{ T_{ij} \} \) and the cost/time matrix \( \{ c_{ij} \} \). Three constraint equations may be presented, considering the trip frequency and trip cost effects.

\[
\begin{align*}
\sum_i p_{ij} &= 1 \quad (1) \\
\sum_i p_{ij} T_{ij} &= R_1 \quad (2) \\
\sum_i p_{ij} c_{ij} &= R_2 \quad (3)
\end{align*}
\]

where \( R_1 \) and \( R_2 \) are constants. \( \{ p_{ij} \} \) is termed the probability bias matrix. Equation (1) is the natural constraint and always holds. Equation (2) is called the trip frequency constraint and equation (3) the trip cost.
constraint. \( R_1 \) is the mean trip cost and \( R_2 \) may be considered the average number of trips between linkages, or the mean trip frequency. Maximising the Shannon measure of entropy \( S \) (where \( S = -\Sigma \Sigma p_i^o \ln p_i^o \)) subject to the above constraints, the Lagrangian function \( L \) is obtained as given by Equation (4).

\[
L = -\Sigma \Sigma p_i^o \ln p_i^o + (\Sigma \Sigma p_i^o - 1) + \alpha (\Sigma \Sigma p_i^o T_i^o - R_1) + \beta (\Sigma \Sigma p_i^o c_i^o - R_2)
\]

\[
\Rightarrow (\delta L/\delta p_i^o) = -1 - \ln p_i^o + 1 + \alpha T_i^o + \beta c_i^o = 0 \quad (5)
\]

\[
\Rightarrow p_i^o = \exp(\alpha T_i^o + \beta c_i^o)
\]

\[
\alpha^o \text{ may be considered a parameter associated with the number of trips between zones } i \text{ and } j \text{ per unit trip interaction between zones } i \text{ and } j, \text{ or the rate of trip-making between zones } i \text{ an } j, \text{ or the trip frequency parameter. } \beta^o \text{ is a parameter associated with the number of trips between } i \text{ and } j \text{ per unit cost between zones } i \text{ and } j, \text{ or the trip cost parameter.}
\]

3. Model Calibration

The maximum entropy distribution represents the most unbiased, most random and most objective distribution consistent with the constraints (Kapur and Kesavan, 1987, p.36), and there is always a family of most unbiased probability distributions (ibid., p.192) which results in a range of entropy values for the number of constraints chosen. It is, thus, expected that there may be many combinations of the two parameters satisfying the natural constraint (i.e., Equation (1)). Therefore this is used as only the first criterion. The second criterion is the selection of the probability distribution with the highest entropy value in order to identify the member of the family which gives the best knowledge of the system while maintaining the highest randomness. Equation (7) may be restructured as follows:

\[
p_i j o p i o = \alpha o p_i j o T_i o + \beta o p_i j o c_i o \quad (8)
\]

\[
\Rightarrow -\Sigma \Sigma j p_i j o p i o = -\alpha o \Sigma \Sigma j p_i j o T_i o - \beta o \Sigma \Sigma j p_i j o c_i o
\]

\[
\Rightarrow S = -\alpha o R_1 - \beta o R_2 \quad (9)
\]

The entropy of the distribution \( S \) is given by Equation (10). A multi-parameter direct unconstrained search routine is used to find \( \alpha^o \) and \( \beta^o \) by (a) satisfying the natural constraint, and (b) simultaneously maximising the Entropy value, \( S \).

The difficulty of calibrating a two-parameter model has been described in detail (Batty, 1976), and it was suggested that a best-fit is only obtained when good initial estimates of the parameters are known. Since no initial parameters are known in this case the model is calibrated several times, with different combinations of initial parameter estimates and increment values on each run.

The datasets have been used to calibrate the new model, together with a traditionally doubly constrained gravity model with an exponential cost deterrent. Webber (1984) remarked about the peculiarity of using non-directional statistics to assess spatial models. The plots of trip residual patterns have therefore been used to give spatial evaluation of the performance of these models.

The fit of the new model is perfect and it has been able to capture the data structure exactly. The ultimate goal, however, is accuracy in forecasting. The uniform entropy is the maximum entropy without constraints and is given by the natural logarithm of the total number of trip cells (that is, \( \ln (N x N - N) \)), where \( N \) is the number of zones, since the all the intrazonal linkages were eliminated. The uniform entropy value for a 14-zone OD system is 5.2040.

4. Forecast and the New Model

The standard data given in the horizon year is the origin trip-end \( \{O_i^F\} \), the destination trip-end \( \{D_j^F\} \) and the cost \( \{c_{ij}^F\} \) matrices. The horizon year probability bias matrix may be formed from the following set of constraints:

\[
\Sigma \Sigma p_i^F \ln(O_i^F B_i^F) = Q_1, \quad (11)
\]

\[
\Sigma \Sigma p_j^F \ln(A_j^F D_j^F) = Q_2, \quad (12)
\]

\[
\Sigma \Sigma p_i^F c_{ij}^F = Q_3, \quad (13)
\]

where \( Q_1, Q_2, \text{ and } Q_3 \) are constants, in addition to the natural constraint (Equation (1)). \( A_j^F \) is proposed as an origin inaccessibility factor for zone \( i \), or the inaccessibility of zone \( i \) to all other zones. \( B_i^F \) is proposed as a destination inaccessibility factor for zone \( j \), or the inaccessibility of zone \( j \) from all other zones.

\[
A_i^F = 1.0/\left[\Sigma D_j^F \exp(\gamma c_{ij}^F)\right], \quad (14)
\]

\[
B_j^F = 1.0/\left[\Sigma O_i^F \exp(\theta c_{ij}^F)\right], \quad (15)
\]

The format of the model constraints (Equations 11 and 12) were made so as to produce a trip distribution model similar to the traditional entropy maximising derivation of the gravity model, but without its cumbersome balancing mechanism. The resultant equation is

\[
p_i^F = A_i^F B_i^F O_i^F D_i^F \exp(\beta c_{ij}^F)
\]

where \( \gamma^F \) and \( \theta^F \) are parameters which collectively should assist in determining the trip frequency effect. \( \beta^F \) is the trip cost parameter in the horizon year. These parameters are to be calibrated to the two criteria discussed earlier, where the Entropy \( S \) is given by Equation (17).

\[
S = -\Sigma \Sigma p_i^F \ln p_i^F \quad (17)
\]

Finally, the future zonal trips are computed from Equation (18).

\[
T_{ij}^F = p_i^F T
\]

Where \( T \) is the total number of trips.

5. Results

In preparing a forecast with the gravity model, a new value for the deterrence parameter must always be
determined. For a projected given set of \( \{O_i\} \), \( \{D_j\} \) and \( \{c_{ij}\} \) matrices, the mean trip length is then plotted versus \( \beta \). In order to forecast, the usual procedure is to assume the future mean trip length for this new period and select the appropriate value from the plotted curve. The best forecast fit for each current dataset occurs with the \( \beta \) value when its dataset was calibrated. Any other \( \beta \) further away from this value will give increasingly worse fit. In other words, in order to forecast with the gravity model, it is necessary to have a base year dataset to determine the base year mean trip length; \( \beta^F \) is estimated from determination of the future mean trip length from a plot of the projected mean trip lengths versus \( \beta^F \) values.

Figure 1 shows the estimated trips minus observed trips versus zone linkages for a doubly-constrained gravity model calibration with Trinidad 2006 Data. It should be recalled that this is the best representation by the gravity model, since any shift in the deterrence parameter value at forecast would give worse results. As discussed earlier, there are 182 (i.e., 14x14-14) zone linkages. The total number of persons travelling is 660,500. The deterrence parameter value at calibration is 2.0902.

Figure 1. Doubly-constrained Gravity Model Calibration with Trinidad 2006 Data: Estimated Trips minus Observed Trips versus Zone Linkages

As stated above, the fit of the new entropy model is perfect at calibration, and it has been able to capture the data structure exactly. Figure 2 shows the Probability Bias Model Forecast with Trinidad 2006 Data, at its optimal fit with respect to variations in entropy values; that is, from a visual comparison of the residuals for each entropy value selected.

This is forecast residuals of the new entropy model at optimal entropy fit very much resemble the calibrated gravity residuals, and the highest residuals are generally of the order of 5,000 person trip linkages in zones with average trip end of 68,000. The optimum entropy value was 4.5477. The calibrated parameter values for \( \gamma^F \), \( \theta^F \) and \( \beta^F \) are -0.8350, -0.8300, and -1.9676, respectively.

It is now examined what happens to the forecast by the new entropy when higher and lower entropy values are selected. The entropy value selected was 3.8030. The parameter values for \( \gamma^F \), \( \theta^F \) and \( \beta^F \) are -1.9490, -1.9490, and -5.1486, respectively. Figure 2 shows the Probability Bias Model Forecast with Trinidad 2006 Data, with a lower than optimal entropy value. There a many large overestimated residuals.

Figure 2. Probability Bias Model Forecast with Trinidad 2006 Data: Estimated Trips minus Observed Trips versus Zone Linkages – Optimal Fit Entropy Value

Figure 3 shows the Probability Bias Model Forecast with Trinidad 2006 Data, with a higher than optimal entropy value. The higher entropy value selected was 4.8044. The calibrated parameter values for \( \gamma^F \), \( \theta^F \) and \( \beta^F \) are -0.4342, -0.4342, and -1.0058, respectively. There are many large underestimated residuals.

Figure 3. Probability Bias Model Forecast with Trinidad 2006 Data: Estimated Trips minus Observed Trips versus Zone Linkages – Higher Entropy Value

Figure 4 shows the Estimated Trips minus Observed Trips versus Zone Linkages for a Doubly-Constrained Gravity Model Calibration with Trinidad 1981 Data. The total number of persons travelling is 365,000. The deterrence parameter value at calibration is 2.8819.

Figure 4. Doubly-constrained Gravity Model Calibration with Trinidad 1981 Data: Estimated Trips minus Observed Trips versus Zone Linkages

Figure 5 shows the Probability Bias Model Forecast with Trinidad 1981 Data, at its optimal fit with respect to variations in entropy values; that is, from a visual comparison of the residuals for each entropy value selected.

This is forecast residuals of the new entropy model at optimal entropy fit very much resemble the calibrated gravity residuals, and the highest residuals are generally
of the order of 5,000 person trip linkages in zones. The optimum entropy value was 4.1830. The calibrated parameter values for $\gamma^F$, $\theta^F$ and $\beta^F$ are -0.5935, -0.5935, and -1.4066, respectively.

Figure 5. Probability Bias Model Forecast with Trinidad 1981 Data: Estimated Trips minus Observed Trips versus Zone Linkages – Optimal Fit Entropy Value

It is now examined what happens to the forecast by the new entropy when higher and lower entropy values are selected. The lower entropy value selected was 3.4644. The calibrated parameter values for $\gamma^F$, $\theta^F$ and $\beta^F$ are -1.6091, -1.6083, and -4.2519, respectively. Figure 6 shows the Probability Bias Model Forecast with Trinidad 1981 Data, with a lower optimal entropy value. There are more large overestimated residuals.

Figure 6. Probability Bias Model Forecast with Trinidad 1981 Data: Estimated Trips Minus Observed Trips versus Zone Linkages – Lower Entropy Value

Figure 7 shows the Probability Bias Model Forecast with Trinidad 1981 Data, with a higher than optimal entropy value. The higher entropy value selected was 4.3732. The calibrated parameter values for $\gamma^F$, $\theta^F$ and $\beta^F$ are -0.2928, -0.2933, and –0.7066, respectively. There are more large underestimated residuals.

Figure 7. Probability Bias Model Forecast with Trinidad 1981 Data: Estimated Trips Minus Observed Trips versus Zone Linkages – Higher Entropy Value

6. Conclusion and Recommendations

All transportation planning and traffic micro-simulation software include a module consisting of OD matrix estimation, or a trip distribution model. The most popular OD matrix estimation model is the gravity model. In spite of the large amount of theoretical and empirical work conducted during the period of the 1970s to the 1990s, the gravity model still does not provide adequate explanations of observed patterns of spatial interaction of the model. The literature review presented confirms the continued popularity of the gravity model. However, because of the continued development and application of transportation planning and traffic micro-simulation software, and attempt has been made to improvement the OD matrix forecast results.

The micro-distribution of trip-making has been redefined, in order to capture the unequal likelihood of individual flows within the linkages. The principle of maximum entropy is used to produce a well-specified model with two deterrent factors, one, a trip frequency function and the other, a trip cost function. Calibration is performed using two criteria: the natural constraint and selection of the probability distribution with the highest entropy value. The calibration process is conducted as part of the forecast procedure, which itself is undertaken through the introduction of available information in the form of linear constraints into the probability formulation. Therefore, base year calibration of the model is not required, and travel demand forecasts can be made with only projected information, and in the absence of historical or baseline data.

The projected information given is manipulated to produce the best probability distribution without assuming anything not given. The new entropy model does not require base year data, but is able to apply projected trip end data and travel linkage costs to estimate future travel demands, in accordance with the entropy theory of maximising what is known. The entropy values selected should always tend to higher values as part of the entropy maximising principle, and several resultant OD matrices applied for scenario-testing in the planning process.

Non-spatial statistics have not been applied to assess these spatial models; the plots of trip residual patterns have been used to give spatial evaluation of the performance of these models.

Hawking (2017, page 10) commented that “a theory is a good theory if it satisfies two requirements. It must accurately describe a large class of observations on the basis of a model that contains only a few arbitrary elements, and it must make definite predictions about the results of future observations.” The focus of this paper has been on the improvement of transportation planning OD model forecasts and the associated theory. Only two datasets have been tested in this paper. It would be useful to apply this theory to other datasets of other jurisdictions. It is postulated that any ideas on constraints in the form of
neighbourhood zone linkage effects and the primary urban centre effects would improve the model forecasts.

References:


Parker, C.C. and Parsons-Brinckerhoff (1966), National Transportation Plan Trinidad and Tobago, Government of Trinidad and Tobago.


Author’s Biographical Notes:

Rae J. Furlonge is the Managing Director and Principal of L. F. Systems Limited, Trinidad and Tobago. He has been involved in professional practice in transportation planning and traffic engineering for a long time in both the public and private sectors within Trinidad and Tobago and the wider Caribbean Region. Dr. Furlonge is primarily engaged in traffic impact assessment studies, transportation planning studies with priority on travel demand management, and traffic incident expert witness contributions.